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SOLVING FUZZY NONLINEAR PROGRAMMING PROBLEMS WITH EXTERIOR PENALTY FUZZY VALUED FUNCTIONS

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Abstract. In this article, we explore the exterior penalty fuzzy valued function method for Fuzzy Nonlinear Programming Problems (FNLPP). We present methods of penalty fuzzy valued function for solving constrained optimization problems by converting to unconstrained optimization problems. Triangular fuzzy numbers are used to represent the problem's decision variables and coefficients. Using a new fuzzy arithmetic and fuzzy ranking method on the parametric form of triangular fuzzy numbers, the optimal solution of the FNLPP is found by applying the exterior penalty fuzzy valued function method without changing to its equivalent crisp form. We present a numerical example of the suggested method and compare the results to those produced by existing methods.

Keywords: Triangular fuzzy number, fuzzy arithmetic, unconstrained optimization, nonlinear programming problem, penalty function, fuzzy exterior penalty method.

AMS Subject Classification: 03B52, 03E72, 90C70, 90C30.

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1 Introduction

Traditional optimization techniques have been used successfully for many years. Due to many reasons, real world problems involves uncertainties and inexactness. Hence to formulate and to solve real world problems, the traditional mathematical tools are inefficient. Zadeh (1965) introduced the concept of fuzzy set and it plays a crucial role in solving the real world problems. There after, Bellman et al. (1970) have discussed the concept of decision making in fuzzy nature. Zimmermann (2001) discussed fuzzy set theory and its applications. There are several fuzzy nonlinear production planning and scheduling issues in many real-world situations, such as in industrial planning. Due to the existence of inaccurate information, research on optimization and modeling for nonlinear programming in fuzzy environment is crucial for the development of fuzzy optimization theory and having a wide range of applications to a variety of practical conflicts in the real world.

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Akrami et al. (2016) focused on solving fuzzy nonlinear optimization problems. They treated all coefficients in the objective function and constraints as fuzzy numbers. They transformed the fuzzy problem into a crisp form using α-cuts. This crisp form becomes an interval nonlinear programming problem, which no longer requires the use of membership functions for solving and obtained the interval solution. Al-Naemi (2022) discussed about a new parameter β_k^{Gh} based on the memoryless self-scale DFP QN method in this article. Any line search will suffice to ensure adequate descent property, and also demonstrated that the Zoutendijk condition holds and that the method is globally convergent by using some step-length technique. Behera and Nayak (2012) provided an improved solution to the issues of fuzzy nonlinear programming problems (FNLPP) with linear constraints. They used the Langrangian method and KKT conditions. Cheng (2018) developed an exact and smooth penalty function to transform nonlinear programming problems into unconstrained optimization models. The results indicate that this new penalty function is a reasonable and effective approach for solving a certain class of NLPP.

Cui et al. (2017) provided unconstrained and constrained optimization problems with a focus on communications, networking, and signal processing. Eiselt and Sandblom (2022) treated nonlinear programming as a generalization of linear programming. Another important distinction between linear and nonlinear programming is that in nonlinear programming, constraints are not necessarily needed to ensure finite optimal as in the case of linear programming. Hassan and Baharum (2019) introduced a new penalty function known as the logarithmic penalty function (LPF) and assesses how well the LPF method converges, and a new penalty function method which transformed non-linear constrained optimization with equality constraints into an unconstrained optimization problem. Jameel and Radhi (2014) developed penalty function method and mixed with Nelder and Mend's algorithm of direct optimization problem to solve FNLPP. Jayswal and Arana-Jimenez (2022) presented their findings on optimizing control problems involving first-order partial differential equations and data uncertainty, known as MCOPU. They derived the robust sufficient optimality conditions for (MCOPU) under the convexity hypotheses and then modulate the unconstrained control problem via the absolute value penalty function method and establish the equivalence between the robust solutions set of the constrained and unconstrained control problems.

Kemal (2017) discussed interior and exterior penalty methods for finding optimal solutions of nonlinear optimization problems by reducing to unconstrained optimization problems. Lavezzi et al. (2022) proposed a set of guidelines to select a solver for the solution of nonlinear programming problems, and comparison of the convergence performances of commonly used solvers for both unconstrained and constrained nonlinear programming problems. Lu and Mei (2023) studied a type of bi-level optimization problem that includes both unconstrained and constrained optimization. In these problems, the lower-level portion is a convex optimization problem, while the upper-level part may involve non-convex optimization. Then they developed penalty methods for solving them, whose subproblems turn out to be a structured minimax problem and are suitably solved by a first-order method developed. Micheal et al. (2021) proposed a method in such a way that mixed (involving equality and inequality) constraint problems can be solved efficiently by the exterior penalty function (EPF) method and robustly and also its performance (in terms of convergence rate and accuracy) is compared with the standard ensemble-based optimization (EnOpt) method. Na et al. (2023) developed an active-set stochastic sequential quadratic programming (StoSQP) algorithm that utilizes a differentiable exact augmented Lagrangian as the merit function. The algorithm adaptively selects the penalty parameters of the augmented Lagrangian, and performs a stochastic line search to decide the step size.

Nagoorgani and Sudha (2019) discussed optimality conditions for fuzzy non-linear unconstrained minimization problems. The cost coefficients are represented by triangular fuzzy numbers and presented some numerical examples. Panigrahi et al. (2022) have converted the fuzzy nonlinear system of equations into an unconstrained fuzzy multivariable optimization problem with preserving the operating constraints. They developed a fuzzy inner-outer direct search method and obtained uncertain solutions. Salman (2021) introduced penalty function methods for solving optimization problems with constraints. The methods they discussed aim to transform a constrained optimization problem into an unconstrained one. After this transformation, they apply standard search techniques like the exterior penalty function method and the interior penalty method to find solutions. Sharma et al. (2021, 2022) have introduced a new approach for addressing multi-objective aspirational level fractional transportation problems involving fuzzy parameters. They also proposed a Fermatean fuzzy ranking function in optimization of intuitionistic fuzzy transportation problems.

Uma Maheswari and Ganesan (2019) proposed a fuzzy version of the Kuhn-Tucker condition for fully fuzzy nonlinear programming problems and found their optimal fuzzy solutions. They used the Gradient method (also known as the Steepest Descent Method of Cauchy) to convert it into an unconstrained multi-variable fuzzy optimization problem. Vanaja and Ganesan (2024) proposed an interior fuzzy penalty function method for solving fuzzy nonlinear programming problems. Wang and Zhu (2016) proposed conjugate gradient path method for solving derivative-free unconstrained optimization. The iterative direction is obtained by constructing and solving quadratic interpolation model of the objective function with conjugate gradient methods. Yamakawa et al. (2023) discussed a new nonlinear optimization model to solve semi definite optimization problems (SDPs), providing some properties related to local optimal solutions. Yuan et al. (2023) proposed a two-phase constraint-handling technique is integrated into the evolutionary algorithms to solve constrained optimization problems (called TPDE). A constrained optimization EAs (COEA) based on the two-phase CHT (TPDE), which consists of the exploration phase and exploitation phase, and the EPM and IPM are utilized for selection.

Zangwill (1967) introduced Non-Linear Programming Via Penalty Functions. The main contribution of this research paper is as follows: Most of the authors‘ have transformed the fuzzy nonlinear programming problems into one or more equivalent crisp nonlinear programming problems and obtained the crisp solution. By using a new fuzzy arithmetic and ranking on the parametric form of the triangular fuzzy numbers and by using the exterior penalty fuzzy valued functions method, we obtain the fuzzy optimal solution of the given fuzzy nonlinear programming problems without converting to its‘ equivalent crisp form. We prove a lemma and a convergence theorem for the exterior penalty fuzzy valued functions method. A numerical example is provided to show the efficacy of the proposed method and the results are compared with the existing ones‘. We discuss a real world application of fuzzy nonlinear programming problems in advertising sector. The results of the methods are shown graphically.

2 Preliminaries

Definition 1. *A fuzzy number $\tilde{M}$ is a fuzzy set on R whose membership function $\tilde{M} : R \to [0,1]$ has the following characteristics:*

1. *$\tilde{M}(y)$ is convex, i.e., $\tilde{M}(\lambda y_1 + (1-\lambda)y_2) \geq \min\{\tilde{M}(y_1), \tilde{M}(y_2)\}$, $\lambda \in [0,1]$, for all $y_1, y_2 \in R$.*

2. *$\tilde{M}$ is normal, i.e., there exists an $y \in R$ such that $\tilde{M}(y) = 1$*

3. *$\tilde{M}$ is upper semi-continuous.*

4. *$sup(\tilde{M})$ is bounded in R.*

We use the notation $F(R)$ to denote the set of all fuzzy numbers defined on R.

Definition 2. *A triangular fuzzy number (TFN) $\tilde{M}$ is a fuzzy number $\tilde{M}$ on R whose membership function $\tilde{M} : R \to [0,1]$ has the following characteristics:*

$$\tilde{M}(y) = \begin{cases} \dfrac{y - m_1}{m_2 - m_1}, & m_1 \leq y \leq m_2 \\ \dfrac{m_3 - y}{m_3 - m_2}, & m_2 \leq y \leq m_3 \\ 0, & elsewhere \end{cases}$$

We denote this triangular fuzzy number by $\tilde{M} = (m_1, m_2, m_3)$.

Definition 3. *A fuzzy number* $\tilde{M}$ *can also be represented as a pair* $(\underline{m};\ \overline{m})$ *of functions* $\underline{m}(\beta),\ \overline{m}(\beta),\ 0 \leq \beta \leq 1$ *which satisfy the following requirements:*

1. $\underline{m}(\beta)$ *is a bounded monotonic increasing left continuous function.*
2. $\overline{m}(\beta)$ *is a bounded monotonic decreasing left continuous function.*
3. $\underline{m}(\beta) \leq \overline{m}(\beta), 0 \leq \beta \leq 1$.

Definition 4. *(***Parametric Form***)*
Let $\tilde{M} = (m_1, m_2, m_3)$ *be a triangular fuzzy number and* $\underline{m}(\beta) = m_1 + (m_2 - m_1)\beta,\ \overline{m}(\beta) = m_3 - (m_3 - m_2)\beta,\ \beta \in [0, 1]$. *The parametric form of the TFN is defined as* $\tilde{M} = (m_0, m_*, m^*)$, *where* $m_* = m_0 - \underline{m}$ *and* $m^* = \overline{m} - m_0$ *are the left and right fuzziness index functions respectively. The number* $m_0 = \left(\dfrac{\underline{m}(1) + \overline{m}(1)}{2}\right)$ *is called the location index number. When* $\beta = 1$, *we get* $m_0 = m_2$.

2.1 Arithmetic Operations on Fuzzy Numbers

Ma et al. (1999) have expressed all the fuzzy numbers in their parametric form, i.e. in the form of location index and fuzziness index functions. They proposed a new fuzzy arithmetic operation on which the location index number follows the usual arithmetic and the fuzziness index functions are following the lattice rule which is least upper bound and greatest lower bound in the lattice L. That is for $m, n \in L,\ m \vee n = \max\{m, n\}$ and $m \wedge n = \min\{m, n\}$. For any two fuzzy numbers $\tilde{M} = (m_0, m_*, m^*),\ \tilde{N} = (n_0, n_*, n^*)$ and $* \in \{+, -, \times, \div\}$, the arithmetic operations are defined as

$$\begin{aligned} \tilde{M} * \tilde{N} =& (m_0, m_*, m^*) * (n_0, n_*, n^*) = (m_0 * n_0, m_* \vee n_*, m^* \vee n^*) \\ =& (m_0 * n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\}) \end{aligned}$$

In particular for $\tilde{M} = (m_0, m_*, m^*)$ and $\tilde{N} = (n_0, n_*, n^*)$ in $F(R)$, we have

1. Addition :$\tilde{M} + \tilde{N} = (m_0, m_*, m^*) + (n_0, n_*, n^*) = (m_0 + n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\})$
2. Subtraction :$\tilde{M} - \tilde{N} = (m_0, m_*, m^*) - (n_0, n_*, n^*) = (m_0 - n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\})$
3. Multiplication :$\tilde{M} \times \tilde{N} = (m_0, m_*, m^*) \times (n_0, n_*, n^*) = (m_0 \times n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\})$
4. Division :$\tilde{M} \div \tilde{N} = (m_0, m_*, m^*) \div (n_0, n_*, n^*) = (m_0 \div n_0, \max\{m_*, n_*\}, \max\{m^*, n^*\})$, provided $n_0 \neq 0$.

2.2 Ranking of Fuzzy Numbers

Ranking of fuzzy numbers plays a major role in decision making process under fuzzy environment. Different types of ranking methods suggested by several authors are available in the literature. In this article, we use an efficient ranking method based on the graded mean.

For $\tilde{M} = (m_0, m_*, m^*) \in F(R)$, define $\Re : F(R) \to R$ by $\Re(\tilde{M}) = \left(\dfrac{m_* + 4m_0 + m^*}{6}\right)$.

For any two triangular fuzzy numbers $\tilde{M} = (m_0, m_*, m^*)$ and $\tilde{N} = (n_0, n_*, n^*)$ in $F(R)$, we have the following comparison:

- If $\Re(\tilde{M}) < \Re(\tilde{N})$, then $\tilde{M} \prec \tilde{N}$
- If $\Re(\tilde{M}) > \Re(\tilde{N})$, then $\tilde{M} \succ \tilde{N}$
- If $\Re(\tilde{M}) = \Re(\tilde{N})$, then $\tilde{M} \approx \tilde{N}$.

3 Fuzzy Non Linear Programming Problems (FNLPP)

Consider a general fuzzy nonlinear programming problem

$$\begin{aligned} &\min \tilde{f}(y) \\ &\text{subject to } \tilde{h}_i(y) \approx \tilde{0} \text{ for } i = 1, 2, \cdots, l \\ &\tilde{g}_i(y) \preceq \tilde{0} \text{ for } j = 1, 2, \cdots, m, \\ &Y \succeq \tilde{0} \end{aligned} \tag{1}$$

where $\tilde{f}, \tilde{h}_1, \cdots, \tilde{h}_l, \tilde{g}_1, \cdots, \tilde{g}_m$ are continuous fuzzy valued functions defined on R^n.

A vector $\mathbf{Y} = (y_1, y_2, y_3, \cdots, y_n)$ is said to be a feasible solution to the FNLPP if it satisfies the constraints and the non negativity restriction of the FNLPP. The set of all feasible solutions forms the feasible region and is defined by

$$Y = \{\mathbf{y} \in R^n / \tilde{h}_i(y) \approx \tilde{0} \text{ for } i = 1, 2, \cdots, l, \ \tilde{g}_j(y) \preceq \tilde{0} \text{ for } j = 1, 2, \cdots, m \text{ and } \mathbf{y} \succeq \tilde{0}\}.$$

4 Exterior penalty fuzzy valued function methods

Definition 5. *A continuous fuzzy valued function $\tilde{P} : R^n \to F(R)$ is said to be a penalty fuzzy valued function if $\tilde{P}$ satisfies:*
(i). $\tilde{P}(y) \approx \tilde{0}$ if and only if $\tilde{g}_i(y) \preceq \tilde{0}$
(ii). $\tilde{P}(y) \succ \tilde{0}$ otherwise $\tilde{g}_i(y) \npreceq \tilde{0}$

Definition 6. *Consider a FNLPP with inequality constraints*

$$\begin{aligned} &\min \tilde{f}(y) \\ &\textit{subject to } \tilde{g}_i(y) \preceq \tilde{0} \quad \textit{for} \quad i = 1, 2, 3, \cdots, m \\ &Y \succeq \tilde{0}, \end{aligned} \tag{2}$$

where $\tilde{f}, \tilde{g}_1, \cdots, \tilde{g}_m$ are continuous fuzzy valued functions defined on R^n.

Exterior penalty fuzzy valued function method uses penalty fuzzy valued function to penalize the infeasible points but not feasible points. In such methods, every sequence of unconstrained optimization attains an improved yet infeasible solution. These methods are known as exterior penalty methods.

The exterior penalty fuzzy valued function method generally use the auxiliary function ψ as,

$$\tilde{\psi}_\mu(y) = \tilde{f}(y, \mu) = \tilde{f}(y) + \mu\tilde{\alpha}(y) \quad \text{for} \quad y \in R^n,$$

where the penalty fuzzy valued function $\tilde{\alpha}(y)$ is defined by

$$\tilde{\alpha}(y) = \sum_{i=1}^{m} [\max\{0, \tilde{g}_j(y)\}]^p + \sum_{i=1}^{l} |\tilde{h}_i(y)|^p, (i.e.\ p \succeq \tilde{0}) \tag{3}$$

For a positive integer p and a non-negative penalty parameter μ. If y resides within the feasible region, then $\tilde{\alpha}(y) \approx \tilde{0}$ implies no penalty is incurred. Therefore, a penalty is applicable only when the point y is not feasible region, i.e., for a point y such that $\tilde{g}_j(y) \succ \tilde{0}$ for some $j = 1, 2, \cdots, m$ or $\tilde{h}_i(y) \not\approx \tilde{0}$ for some $i = 1, 2, \cdots, l$.

$$\tilde{\psi}_\mu(y) = \tilde{f}(y, \mu) = \tilde{f}(y) + \mu \sum_{i=1}^{m} [\max\{0, \tilde{g}_j(y)\}]^p$$

Concerning the auxiliary function, the impact of the second term on the right side is to augment $\tilde{\psi}\mu(y)$ in correlation with the p^{th} power of the extent to which the constraints are breached. Consequently, a penalty is incurred for constraint violations, and the penalty amount escalates at a greater rate than the extent of constraint violation (for $p \succ 1$).
Let's examine the behavior of $\tilde{\psi}\mu(y)$ for different values of p.
(i). If $\quad p = 0$

$$\begin{aligned}\tilde{\psi}_\mu(y) = \tilde{f}(y, \mu) =& \tilde{f}(y) + \mu \sum_{i=1}^{m} [\max\{0, \tilde{g}_j(y)\}]^0 \\ &= \tilde{f}(y) + m\mu, \quad \text{for all} \quad \tilde{g}_j(y) \succ \tilde{0} \\ &= \tilde{f}(y), \quad \text{for all} \quad \tilde{g}_j(y) \prec \tilde{0}\end{aligned}$$

The function exhibits discontinuity at the boundary of the acceptable region, and hence it will be very difficult to minimize this function.
(ii). If $\quad 0 \leq p \leq 1$
Here the $\tilde{\psi}$-function will be continuous, but the penalty for violating a constraint may be too small. Additionally, the derivatives of the function exhibit discontinuity along the boundary. Consequently, it will be hard to minimize the $\tilde{\psi}$-function.
(iii). If $\quad p = 1$
In this case, under specific constraints, Zangwill (1967) demonstrated that there exists a sufficiently large μ_0 such that the minimum of ψ precisely corresponds to the constrained minimum of the original problem for all $\mu_\kappa \geq \mu_0$. Nevertheless, the contours of the $\tilde{\psi}$-function posses discontinuous first-order derivatives along the boundary. Consequently, despite the convenience of selecting a single μ_κ that achieves the constrained minimum in an unconstrained minimization, the method lacks attractiveness from a computational point of view.
(iv). If $\quad p \geq 1$
The first-order derivatives of the $\tilde{\psi}$-function will be continuous, and they are expressed as:

$$\frac{\partial \tilde{\psi}}{\partial y_i} = \frac{\partial \tilde{f}}{\partial y_i} + \mu_\kappa \sum_{j=1}^{m} p[\max\{0, \tilde{g}_j(y)\}]^{p-1} \frac{\partial \tilde{g}_j(y)}{\partial y_i}.$$

Generally, for the practical computation, the value of p is chosen as 2, and therefore, we will use $p = 2$ in the subsequent discussion of the penalty method with.

$$\tilde{\alpha}(y) = \sum_{i=1}^{m} [\max\{0, \tilde{g}_j(y)\}]^2$$

4.1 Convergence of Exterior Penalty Fuzzy Valued Function Methods

Consider a sequence of values μ_κ with $\mu_\kappa \uparrow \infty$ as $\kappa \to \infty$, and let $\mathbf{y}^\kappa$ be the minimizer of $\tilde{f}_{\mu_\kappa}(y) = \tilde{f}(y) + \mu_\kappa \tilde{\alpha}(y)$ for each κ.

Lemma 1. *Suppose that there exist a feasible optimal solution $\tilde{y}^*$ for FNLPP (2) and let $\tilde{\alpha}$ be a continuous function defined by (3). If for each μ_κ, there is a minimizer $\mathbf{y}^\kappa \in Y$ of $\tilde{f}_{\mu\kappa}(y) = \tilde{f}(y) + \mu_\kappa \tilde{\alpha}(y)$, then following properties holds true for $0 < \mu_\kappa < \mu_{\kappa+1}$.*

(i). $\tilde{f}_{\mu k}(y^{\kappa}) \preceq \tilde{f}_{\mu\kappa+1}(y^{\kappa+1})$

(ii). $\tilde{\alpha}(y^{\kappa}) \succeq \tilde{\alpha}(y^{\kappa+1})$

(iii). $\tilde{f}(y^{\kappa}) \preceq \tilde{f}(y^{\kappa+1})$

(iv). $\tilde{f}(y^{*}) \succeq \tilde{f}_{\mu\kappa}(y^{\kappa}) \succeq \tilde{f}(y^{\kappa})$

Proof. (i). Since $0 < \mu_k < \mu_{k+1}$ and $\tilde{\alpha}(y) \succeq \tilde{0}$, we get $\mu_\kappa\tilde{\alpha}(y^{\kappa+1}) \preceq \mu_{\kappa+1}\tilde{\alpha}(x^{\kappa+1})$
Furthermore, since (y^{κ}) minimizes $\tilde{f}_{\mu\kappa}(y)$, we have

$$\begin{aligned}\tilde{f}_{\mu\kappa}(y^{\kappa}) &= \tilde{f}(y^{\kappa}) + \mu_\kappa\tilde{\alpha}(y^{\kappa})\\ &\preceq \tilde{f}(y^{\kappa+1}) + \mu_\kappa\tilde{\alpha}(y^{\kappa+1})\\ &\preceq \tilde{f}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{\alpha}(y^{\kappa+1})\\ &\approx \tilde{f}_{\mu\kappa+1}(y^{\kappa+1})\end{aligned} \tag{4}$$

$$\Rightarrow \tilde{f}_{\mu\kappa}(y^{\kappa}) \preceq \tilde{f}_{\mu\kappa+1}(y^{\kappa+1}) \tag{5}$$

(ii). As $(y^{\kappa+1})$ minimizes $\tilde{f}_{\mu\kappa+1}(y^{\kappa+1})$, we have

$$\tilde{f}(y^{\kappa+1}) + \mu_{\kappa+1}\tilde{\alpha}(y^{\kappa+1}) \preceq \tilde{f}(y^{\kappa}) + \mu_{\kappa+1}\tilde{\alpha}(y^{\kappa}) \tag{6}$$

Similarly, as y^{κ} minimizes $\tilde{f}_{\mu\kappa}(y)$, we have

$$\tilde{f}(y^{\kappa}) + \mu_\kappa\tilde{\alpha}(y^{\kappa}) \preceq \tilde{f}(y^{\kappa+1}) + \mu_\kappa\tilde{\alpha}(y^{\kappa+1}) \tag{7}$$

Adding (6) and (7) and simplifying, we get

$$[\mu_{\kappa+1} - \mu_\kappa][\tilde{\alpha}(y^{\kappa}) - \tilde{\alpha}(y^{\kappa+1})] \succeq \tilde{0}.$$

Since $\mu_{\kappa+1} \succ \mu_\kappa$, we have $\tilde{\alpha}(y^{\kappa}) - \tilde{\alpha}(y^{\kappa+1}) \succeq \tilde{0} \Rightarrow \tilde{\alpha}(y^{\kappa}) \succeq \tilde{\alpha}(y^{\kappa+1})$.
(iii). From inequality (4) we get

$$\tilde{f}(y^{\kappa}) - \tilde{f}(y^{\kappa+1}) \preceq \mu_\kappa[\tilde{\alpha}(y^{\kappa+1}) - \tilde{\alpha}(y^{\kappa})]. \tag{8}$$

Since, $\tilde{\alpha}(y^{\kappa+1}) - \tilde{\alpha}(y^{\kappa}) \preceq \tilde{0}$ and $\mu_\kappa \succ 0$, we have

$$\tilde{f}(y^{\kappa}) - \tilde{f}(y^{\kappa+1}) \preceq \tilde{0} \;\Rightarrow\; \tilde{f}(y^{\kappa}) \preceq \tilde{f}(y^{\kappa+1}).$$

(iv). Suppose that $\mathbf{y}^*$ be an optimum solution, then we have

$$\begin{aligned}\tilde{f}(y^{\kappa}) &\preceq \tilde{f}(y^{\kappa}) + \mu_\kappa\tilde{\alpha}(y^{\kappa})\\ &\preceq \tilde{f}(y^{*}) + \mu_\kappa\tilde{\alpha}(y^{*})\\ &\approx \tilde{f}(y^{*}),\ \text{since } \mu_\kappa\tilde{\alpha}(y^{\kappa}) \succeq \tilde{0} \text{ and } \tilde{\alpha}(y^{*}) \approx \tilde{0}.\end{aligned}$$

□

Theorem 1. *Suppose that there exist a feasible optimal solution $\tilde{y}^*$ for FNLPP (2) and let $\tilde{\alpha}$ be a continuous function defined by (3). Furthermore, for each μ_κ, suppose there exists a solution $\mathbf{y}^{\kappa} \in Y$ that minimizes $\tilde{f}(y) + \mu_\kappa\tilde{\alpha}(y)$ subject to $\mathbf{y} \in Y$, and that $\mathbf{y}^{\kappa}$ is contained in a compact subset of Y. Then, the limit $\bar{y}$ of any convergent subsequence of $\{y^{\kappa}\}$ is an optimal solution to the original problem, and $\mu_\kappa\tilde{\alpha}(y^{\kappa}) \to 0$ as $\mu_\kappa \to \infty$.*

Proof. Let $\bar{y}$ be a limit point of (y^κ). From the continuity of the function involved,

$$\lim_{\kappa\to\infty} \tilde{f}(y^\kappa) = \tilde{f}(\bar{y})$$

Also from (iv) of lemma (1), we have

$$\tilde{f}^*_\mu = \lim_{\kappa\to\infty} \tilde{f}_{\mu\kappa}(y^\kappa) \preceq \tilde{f}(y^*) \text{ and } \lim_{\kappa\to\infty} \tilde{f}(y^\kappa) = \tilde{f}(\bar{y}) \preceq \tilde{f}(y^*) \tag{9}$$

and hence

$$\lim_{\kappa\to\infty} [\tilde{f}_{\mu\kappa}(y^\kappa) - \tilde{f}(y^\kappa)] = \tilde{f}^*_\mu - \tilde{f}(\bar{y})$$

which implies that

$$\lim_{\kappa\to\infty} \mu_\kappa \tilde{\alpha}(y^\kappa) = \tilde{f}^*_\mu - \tilde{f}(\bar{y}) \tag{10}$$

(by continuity of $\tilde{\alpha}$) which is equivalent to

$$\tilde{\alpha}(\bar{y}) = \lim_{\kappa\to\infty} \frac{1}{\mu_\kappa}[\tilde{f}^*_\mu - \tilde{f}(\bar{y})] = 0, \text{ since } \tilde{f}^*_\mu - \tilde{f}(\bar{y}) \text{ is constant and } \frac{1}{\mu_\kappa} \to 0 \text{ as } \kappa \to \infty.$$

Hence, $\bar{y}$ is a feasible solution to the original constrained problem. Given that $\bar{y}$ is feasible and y^* is the minimizer of the original constrained problem,

$$\tilde{f}(y^*) \preceq \tilde{f}(\bar{y}) \tag{11}$$

Hence, from (9) and (11)

$$\tilde{f}(y^*) \approx \tilde{f}(\bar{y})$$

Hence, the sequence (y^κ) converges to the optimal solution of the original constrained problem. From (10), we have $\lim_{\kappa\to\infty} \mu_\kappa \tilde{\alpha}(y^\kappa) = \tilde{f}^*_\mu - \tilde{f}(\bar{y}) \approx \tilde{0}$. Thus, $\mu_\kappa \tilde{\alpha}(y^\kappa) \to 0$ as $\mu_\kappa \to \infty$.

This theorem implies that, as $\mu_\kappa \to \infty$, the optimal solution (y^κ) to $\tilde{f}_{\mu\kappa}(y)$ can be brought arbitrarily close to the feasible region. Although the optimal solutions (y^κ) are typically infeasible, increasing μ_κ results in the generated points approaching an optimal solution from outside the feasible region.

□

4.2 Algorithm for Exterior Penalty Fuzzy Valued Function Method

Step 1. Assume a growth parameter $\gamma > 1$, a stopping parameter (tolerance) $\epsilon > 0$, and an initial value μ_1. Consider (y^κ) as the starting point, violating at least one constraint, and formulate the objective function $\tilde{f}_{\mu\kappa}(y)$ for $\kappa = 1, 2, 3, \cdots, n$.
Step 2. Commencing with (y^κ), apply an unconstrained search technique to identify the point minimizing $\tilde{f}_{\mu\kappa}(y)$, denoting it as $(y^{\kappa+1})$, which then becomes the new starting point.
Step 3. If $\|(y^{\kappa+1}) - (y^\kappa)\| < \epsilon$, or if the difference between two successive objective function values is smaller than ϵ, i.e., $\mid \tilde{f}(y^{\kappa+1}) - \tilde{f}(y^\kappa) \mid \prec \epsilon$, then stop with $(y^{\kappa+1})$ as an estimate of the optimal solution. Otherwise, set $\mu_{\kappa+1} = \gamma\mu_\kappa$, formulate the new $\tilde{f}_{\mu\kappa+1}(y)$, put $\kappa = \kappa + 1$, and return to step 1.
Note: We solve a sequence of problems by progressively increasing the μ values. In other words, for $0 < \mu_\kappa < \mu_{\kappa+1}$, the optimal point (y^κ) for the penalized objective function $\tilde{f}_{\mu\kappa}(x)$, the sub-problem at the κ^{th} iteration becomes the initial point for the subsequent problem, where $\kappa = 1, 2, 3, \cdots, n$. To obtain the optimum (y^κ), we assume that the penalized function has a solution for all positive values of μ_κ.

5 Numerical Example

Consider a FNLPP discussed by Akrami et al. (2016)

$$\begin{aligned} \max \tilde{f}(\tilde{y}) =& (1,3,4)\tilde{y}_1^2 + (1,2,3)\tilde{y}_2^2 \\ \text{subject to } & (0,1,3)\tilde{y}_1 + (2,3,5)\tilde{y}_2 \preceq (3,4,6) \\ & (1,2,4)\tilde{y}_1 - (0,1,2)\tilde{y}_2 \preceq (1,2,5) \\ & \tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}. \end{aligned} \tag{12}$$

Solution: The given maximization FNLPP is converted into and equivalent minimization FNLPP as

$$\begin{aligned} \min \tilde{f}(\tilde{y}) =& -(1,3,4)\tilde{y}_1^2 - (1,2,3)\tilde{y}_2^2 \\ \text{subject to } & (0,1,3)\tilde{y}_1 + (2,3,5)\tilde{y}_2 - (3,4,6) \preceq \tilde{0} \\ & (1,2,4)\tilde{y}_1 - (0,1,2)\tilde{y}_2 - (1,2,5) \preceq \tilde{0} \\ & \tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}. \end{aligned} \tag{13}$$

The parametric form of the given FNLPP (13) is given by

$$\begin{aligned} \min \tilde{f}(\tilde{y}) =& -(3, 2-2\beta, 1-\beta)\tilde{y}_1^2 - (2, 1-\beta, 1-\beta)\tilde{y}_2^2 \\ \text{subject to } \tilde{g}(\tilde{y}_1) =& (1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta) \preceq \tilde{0} \\ \tilde{g}(\tilde{y}_2) =& (2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta) \preceq \tilde{0} \\ & \tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}, \quad \beta \in [0,1]. \end{aligned}$$

Define the penalty function $\tilde{\alpha}(\tilde{y}) = [\max\{\tilde{g}(\tilde{y}), 0\}]^2$. Thus $\tilde{\alpha}_1(\tilde{y}) \approx \tilde{0}$ and $\tilde{\alpha}_2(\tilde{y}) \approx \tilde{0}$, for $\tilde{g}(\tilde{y}) \preceq \tilde{0}$ and

$$\begin{aligned} \tilde{\alpha}_1(\tilde{y}) &= [(1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta)]^2, \\ \tilde{\alpha}_2(\tilde{y}) &= [(2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta)]^2, \end{aligned}$$

for $\tilde{g}(\tilde{y}) \succ \tilde{0}$.
Then the corresponding unconstrained fuzzy optimization problem is

$$\tilde{f}_{\mu\kappa}(\tilde{y}) = -(3, 2-2\beta, 1-\beta)\tilde{y}_1^2 - (2, 1-\beta, 1-\beta)\tilde{y}_2^2 + \mu_\kappa \tilde{\alpha}_1(\tilde{y}) + \mu_\kappa \tilde{\alpha}_2(\tilde{y})$$

Case:1 If $\tilde{\alpha}_1(\tilde{y}) \approx \tilde{0}$ and $\tilde{\alpha}_2(\tilde{y}) \approx \tilde{0}$, for $\tilde{g}(\tilde{y}) \preceq \tilde{0}$, then the optimal solution to

$$\min \ \tilde{f}_{\mu\kappa}(\tilde{y}) = -(3, 2-2\beta, 1-\beta)\tilde{y}_1^2 - (2, 1-\beta, 1-\beta)\tilde{y}_2^2 + \mu_\kappa \tilde{\alpha}_1(\tilde{y}) + \mu_\kappa \tilde{\alpha}_2(\tilde{y})$$

is at $y^* = (0,0)$ and is infeasible.
Case:2 If $\tilde{\alpha}_1(\tilde{y}) = [(1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta)]^2$ and $\tilde{\alpha}_2(\tilde{y}) = [(2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta)]^2$, for $\tilde{g}(\tilde{y}) \succ \tilde{0}$, then

$$\begin{aligned} \min \ \tilde{f}_{\mu\kappa}(\tilde{y}) =& -(3, 2-2\beta, 1-\beta)\tilde{y}_1^2 - (2, 1-\beta, 1-\beta)\tilde{y}_2^2 \\ & + \mu_\kappa((1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta))^2 \\ & + \mu_\kappa((2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta))^2 \end{aligned} \tag{14}$$

The necessary condition for $\tilde{\mathbf{y}}$ to be optimal for (14) implies $\bigtriangledown \tilde{f}_{\mu\kappa}(\tilde{y}) \approx \tilde{0}$. Hence we have

$$\begin{aligned} &\frac{\partial \tilde{f}_{\mu\kappa}}{\partial \tilde{y}_1} = -(6, 2-2\beta, 1-\beta)\tilde{y}_1 + (2, 1-\beta, 2-2\beta)\mu_\kappa \\ &[(1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta)] + (4, 1-\beta, 2-2\beta)\mu_\kappa \\ &[(2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta)] \approx \tilde{0}. \end{aligned}$$

$$\frac{\partial \tilde{f}_{\mu\kappa}}{\partial \tilde{y}_2} = -(4, 1-\beta, 1-\beta)\tilde{y}_2 + (6, 1-\beta, 2-2\beta)\mu_\kappa$$
$$[(1, 1-\beta, 2-2\beta)\tilde{y}_1 + (3, 1-\beta, 2-2\beta)\tilde{y}_2 - (4, 1-\beta, 2-2\beta)] - (2, 1-\beta, 1-\beta)\mu_\kappa$$
$$[(2, 1-\beta, 2-2\beta)\tilde{y}_1 - (1, 1-\beta, 1-\beta)\tilde{y}_2 - (2, 1-\beta, 3-3\beta)] \approx \tilde{0}.$$

This implies that,

$$\begin{aligned} &-(6, 2-2\beta, 1-\beta)\tilde{y}_1 + (10, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_1 + (2, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_2 - (16, 1-\beta, 3-3\beta)\mu_\kappa \approx \tilde{0} \\ &-(4, 1-\beta, 1-\beta)\tilde{y}_2 + (2, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_1 + (20, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_2 - (20, 1-\beta, 3-3\beta)\mu_\kappa \approx \tilde{0} \end{aligned} \tag{15}$$

Solving these equations (15), we get,

$$\tilde{y}_1 = \frac{(8, 1-\beta, 3-3\beta)\mu_\kappa - (1, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_2}{(5, 1-\beta, 2-2\beta)\mu_\kappa - (3, 2-2\beta, 1-\beta)}$$
$$\tilde{y}_2 = \frac{(10, 1-\beta, 3-3\beta)\mu_\kappa - (1, 1-\beta, 2-2\beta)\mu_\kappa\tilde{y}_1}{(10, 1-\beta, 2-2\beta)\mu_\kappa - (2, 1-\beta, 1-\beta)}$$

$$\tilde{\mathbf{y}}^\kappa \approx \Bigg\{ \frac{(70, 1-\beta, 3-3\beta)\mu_\kappa^2 - (16, 1-\beta, 3-3\beta)\mu_\kappa}{(49, 1-\beta, 2-2\beta)\mu_\kappa^2 - (40, 2-2\beta, 2-2\beta)\mu_\kappa + (6, 2-2\beta, 1-\beta)}, \\ \frac{(210, 1-\beta, 3-3\beta)\mu_\kappa^3 - (192, 2-2\beta, 3-3\beta)\mu_\kappa^2 + (30, 2-2\beta, 3-3\beta)\mu_\kappa}{(245, 1-\beta, 2-2\beta)\mu_\kappa^3 - (249, 2-2\beta, 2-2\beta)\mu_\kappa^2 + (70, 2-2\beta, 2-2\beta)\mu_\kappa - (6, 2-2\beta, 1-\beta)} \Bigg\}$$

Let $\mu_{\kappa+1} = \gamma\mu_\kappa$. Since $\gamma > 1$, starting with $\gamma = 10, \mu_1 = 1$ and $\tilde{\mathbf{y}}^1 = (0, 0)$ and using a tolerance of 0.0001 (say), we have the following tables (1),(2),(3),(4).

Table 1: Penalty Iteration Table

κ	μ_κ	$\tilde{y}_1^\kappa$	$\tilde{y}_2^\kappa$
1	1	(3.6,2-2β,3-3β)	(0.8,2-2β,3-3β)
2	10	(1.51798,2-2β,3-3β)	(0.86551,2-2β,3-3β)
3	100	(1.43702,2-2β,3-3β)	(0.85801,2-2β,3-3β)
4	1000	(1.42941,2-2β,3-3β)	(0.85723,2-2β,3-3β)
5	10000	(1.42866,2-2β,3-3β)	(0.85715,2-2β,3-3β)
6	100000	(1.42857,2-2β,3-3β)	(0.85714,2-2β,3-3β)
7	1000000	(1.42857,2-2β,3-3β)	(0.85714,2-2β,3-3β)

Table 2: Continuation to Table 1

κ	$\tilde{g}_1(\tilde{y}^\kappa)$	$\tilde{g}_2(\tilde{y}^\kappa)$
1	(-2,1-β,2-2β)	(-4.4,1-β,3-3β)
2	(-0.114506,1-β,2-2β)	(-0.170442,1-β,3-3β)
3	(-0.011049,1-β,2-2β)	(-0.016028,1-β,3-3β)
4	(-0.001102,1-β,2-2β)	(-0.001594,1-β,3-3β)
5	(-0.00011,1-β,2-2β)	(-0.00017,1-β,3-3β)
6	(-0.000011,1-β,2-2β)	(-0.0000159,1-β,3-3β)
7	(-0.0000011,1-β,2-2β)	(-0.00000159,1-β,3-3β)

Table 3: Continuation to Table 2

κ	$\tilde{\alpha}_1(\tilde{y}^\kappa)$	$\tilde{\alpha}_2(\tilde{y}^\kappa)$	$\mu_\kappa\tilde{\alpha}_1(\tilde{y}^\kappa)$
1	(4.0,1-β,2-2β)	(19.36,1-β,3-3β)	(4,1-β,2-2β)
2	(0.013112,1-β,2-2β)	(0.0290505,1-β,3-3β)	(0.13112,1-β,2-2β)
3	(0.0001221,1-β,2-2β)	(0.0002569,1-β,3-3β)	(0.01221,1-β,2-2β)
4	(0.00001214,1-β,2-2β)	(0.000002541,1-β,3-3β)	(0.001214,1-β,2-2β)
5	(0.000000012,1-β,2-2β)	(0.0000000289,1-β,3-3β)	(0.00012,1-β,2-2β)
6	(0.00000000012,1-β,2-2β)	(0.0000000002581,1-β,3-3β)	(0.000012,1-β,2-2β)
7	(0.0000000000012,1-β,1-β)	(0.0000000002581,1-β,1-β)	(0.0000012,1-β,2-2β)

Table 4: Continuation to Table 3

κ	$\mu_\kappa\tilde{\alpha}_2(\tilde{y}^\kappa)$	$\tilde{f}(\tilde{y}^\kappa)$	$f_{\mu\kappa}(\tilde{y}^\kappa)$
1	(19.36,1-β,3-3β)	(-40.16,2-2β,1-β)	(-16.8,2-2β,3-3β)
2	(0.290505,1-β,3-3β)	(-8.410969,2-2β,1-β)	(-7.989344,2-2β,3-3β)
3	(0.02569,1-β,3-3β)	(-7.667433,2-2β,1-β)	(-7.629533,2-2β,3-3β)
4	(0.002541,1-β,3-3β)	(-7.599343,2-2β,1-β)	(-7.595588,2-2β,3-3β)
5	(0.000289,1-β,3-3β)	(-7.5926204,2-2β,1-β)	(-7.5922104,2-2β,3-3β)
6	(0.00002581,1-β,3-3β)	(-7.591815,2-2β,1-β)	(-7.5917776,2-2β,3-3β)
7	(0.000002581,1-β,3-3β)	(-7.591815,2-2β,1-β)	(-7.59156119,2-2β,3-3β)

From the above tables, we see that the exterior penalty fuzzy valued function method converges at the 7^{th} iteration. Hence the optimal solution of the given FNLPP (12) is $\tilde{y}_1 = (1.42857, 2-2\beta, 3-3\beta)$, $\tilde{y}_2 = (0.85714, 2-2\beta, 3-3\beta)$ with $\max \tilde{f}(\tilde{y}) = (7.59156119, 2-2\beta, 3-3\beta)$. That is the optimal solution of the fuzzy nonlinear programming problem (12) is $\tilde{y}_1 = (-0.57143+2\beta, 1.42857, 4.42857-3\beta)$, $\tilde{y}_2 = (-1.14286+2\beta, 0.85714, 3.85714-3\beta)$ with $\tilde{f}(\tilde{y}) = (5.59156119+2\beta, 7.59156119, 10.5915612-3\beta)$.

6 An application in advertising sector

Himaja and Co. wishes to plan its advertising strategy for which there are two media under consideration, namely Raja Cable and Srija Channel. Both Raja Cable and Srija Channel have a reach of potential customers as the square of the number of appearances. The cost per appearance of one minute is approximately $Rs.6,000$ and $Rs.9,000$ in Raja cable and Srija channel respectively. The budget of Himaja is approximately $Rs.80,000$ per month. There is an important requirement that the total reach for the income group under $Rs.60,000$ per annum should not exceed around $3,000$ potential customers. The reach in Raja Cable and Srija Channel for this income group is approximately 300 and 150 potential customers. How many appearances of one minute advertisements should Himaja plan so as to maximize the total reach?
Solution: We formulate this as a fuzzy nonlinear programming problem as

$$\begin{aligned} \max \tilde{f}(\tilde{y}) =& \tilde{y}_1^2 + \tilde{y}_2^2 \\ \text{subject to } & \tilde{6}\tilde{y}_1 + \tilde{9}\tilde{y}_2 \preceq \tilde{80} \\ & \tilde{2}\tilde{y}_1 + \tilde{y}_2 \preceq \tilde{20} \\ & \tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}. \end{aligned} \tag{16}$$

We assume that all the decision parameters and the coefficients as triangular fuzzy numbers. Then the FNLPP (16) becomes

$$\begin{aligned} \max \tilde{f}(\tilde{y}) =&(0,1,2)\tilde{y}_1^2 + (0,1,2)\tilde{y}_2^2 \\ \text{subject to } &(5,6,7)\tilde{y}_1 + (8,9,10)\tilde{y}_2 \preceq (70,80,90) \\ &(1,2,3)\tilde{y}_1 + (0,1,2)\tilde{y}_2 \preceq (10,20,30) \\ &\tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}. \end{aligned} \tag{17}$$

This maximization FNLPP (17) is converted into and equivalent minimization FNLPP as

$$\begin{aligned} \min \tilde{f}(\tilde{y}) =& -(0,1,2)\tilde{y}_1^2 - (0,1,2)\tilde{y}_2^2 \\ \text{subject to } &(5,6,7)\tilde{y}_1 + (8,9,10)\tilde{y}_2 - (70,80,90) \preceq \tilde{0} \\ &(1,2,3)\tilde{y}_1 + (0,1,2)\tilde{y}_2 - (10,20,30) \preceq \tilde{0} \\ &\tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}. \end{aligned} \tag{18}$$

The parametric form of the FNLPP (18) is given by

$$\begin{aligned} \min \tilde{f}(\tilde{y}) =& -(1,1-\beta,1-\beta)\tilde{y}_1^2 - (1,1-\beta,1-\beta)\tilde{y}_2^2 \\ \text{subject to } \tilde{g}(\tilde{y}_1) =&(6,1-\beta,1-\beta)\tilde{y}_1 + (9,1-\beta,1-\beta)\tilde{y}_2 - (80,10-10\beta,10-10\beta) \preceq \tilde{0} \\ \tilde{g}(\tilde{y}_2) =&(2,1-\beta,1-\beta)\tilde{y}_1 + (1,1-\beta,1-\beta)\tilde{y}_2 - (20,10-10\beta,10-10\beta) \preceq \tilde{0} \\ &\tilde{y}_1, \tilde{y}_2 \succeq \tilde{0}, \quad \beta \in [0,1]. \end{aligned} \tag{19}$$

The FNLPP (19) is transformed in to an unconstrained fuzzy optimization problem as

$$\begin{aligned} \tilde{f}_{\mu\kappa}(\tilde{y}) =& -(1,1-\beta,1-\beta)\tilde{y}_1^2 - (1,1-\beta,1-\beta)\tilde{y}_2^2 \\ &+ \mu_\kappa((6,1-\beta,1-\beta)\tilde{y}_1 + (9,1-\beta,1-\beta)\tilde{y}_2 - (80,10-10\beta,10-10\beta))^2 \\ &+ \mu_\kappa((2,1-\beta,1-\beta)\tilde{y}_1 + (1,1-\beta,1-\beta)\tilde{y}_2 - (20,10-10\beta,10-10\beta))^2 \end{aligned} \tag{20}$$

Solving (20), we get

$$\tilde{\mathbf{y}}^\kappa \approx \left\{ \frac{(4800,10-10\beta,10-10\beta)\mu_\kappa^2 - (2080,10-10\beta,10-10\beta)\mu_\kappa}{(576,1-\beta,1-\beta)\mu_\kappa^2 - (488,1-\beta,1-\beta)\mu_\kappa + (4,1-\beta,1-\beta)}, \right.$$
$$\left. \frac{(314880,10-10\beta,10-10\beta)\mu_\kappa^3 - (489280,10-10\beta,10-10\beta)\mu_\kappa^2 + (5920,10-10\beta,10-10\beta)\mu_\kappa}{(94464,1-\beta,1-\beta)\mu_\kappa^3 - (81184,1-\beta,1-\beta)\mu_\kappa^2 + (1632,1-\beta,1-\beta)\mu_\kappa - (8,1-\beta,1-\beta)} \right\}$$

Let $\mu_{\kappa+1} = \gamma\mu_\kappa$. Since $\gamma > 1$, starting with $\gamma = 10, \mu_1 = 10$ and $\tilde{\mathbf{y}}^1 = (0,0)$ and using a tolerance of 0.0001 (say), we have the following tables (5),(6),(7),(8).

Table 5: Penalty Iteration Table

κ	μ_κ	$\tilde{y}_1^\kappa$	$\tilde{y}_2^\kappa$
1	1	(8.7095,10-10β,10-10β)	(3.0802,10-10β,10-10β)
2	10	(8.3681,10-10β,10-10β)	(3.3100,10-10β,10-10β)
3	100	(8.3368,10-10β,10-10β)	(3.3310,10-10β,10-10β)
4	1000	(8.3337,10-10β,10-10β)	(3.3331,10-10β,10-10β)
5	10000	(8.3334,10-10β,10-10β)	(3.3333,10-10β,10-10β)
6	100000	(8.3333,10-10β,10-10β)	(3.3333,10-10β,10-10β)

Table 6: Continuation to Table 5

κ	$\tilde{g}_1(\tilde{y}^\kappa)$	$\tilde{g}_2(\tilde{y}^\kappa)$
1	(-0.0212,10-10β,10-10β)	(0.4992,10-10β,10-10β)
2	(-0.0014,10-10β,10-10β)	(0.0462,10-10β,10-10β)
3	(-0.0002,10-10β,10-10β)	(0.0046,10-10β,10-10β)
4	(-0.0001,10-10β,10-10β)	(0.0005,10-10β,10-10β)
5	(0.0001,10-10β,10-10β)	(0.0001,10-10β,10-10β)
6	(-0.0005,10-10β,10-10β)	(-0.0001,10-10β,10-10β)

Table 7: Continuation to Table 6

κ	$\tilde{\alpha}_1(\tilde{y}^\kappa)$	$\tilde{\alpha}_2(\tilde{y}^\kappa)$	$\mu_\kappa\tilde{\alpha}_1(\tilde{y}^\kappa)$
1	(0.00045,10-10β,10-10β)	(0.249201,10-10β,10-10β)	(0.0045,10-10β,10-10β)
2	(0.00000196,10-10β,10-10β)	(0.00213444,10-10β,10-10β)	(0.000196,10-10β,10-10β)
3	(0.00000004,10-10β,10-10β)	(0.00002116,10-10β,10-10β)	(0.00004,10-10β,10-10β)
4	(0.00000001,10-10β,10-10β)	(0.00000025,10-10β,10-10β)	(0.0001,10-10β,10-10β)
5	(0.00000001,10-10β,10-10β)	(0.00000001,10-10β,10-10β)	(0.001,10-10β,10-10β)
6	(0.00000025,10-10β,10-10β)	(0.00000001,10-10β,10-10β)	(0.25,10-10β,10-10β)

Table 8: Continuation to Table 7

κ	$\mu_\kappa\tilde{\alpha}_2(\tilde{y}^\kappa)$	$\tilde{f}(\tilde{y}^\kappa)$	$f_{\mu\kappa}(\tilde{y}^\kappa)$
1	(2.49201,10-10β,10-10β)	(-85.3430,10-10β,10-10β)	(-82.84649,10-10β,10-10β)
2	(0.213444,10-10β,10-10β)	(-80.9812,10-10β,10-10β)	(-80.76756,10-10β,10-10β)
3	(0.02116,10-10β,10-10β)	(-80.5978,10-10β,10-10β)	(-80.57624,10-10β,10-10β)
4	(0.0025,10-10β,10-10β)	(-80.5601,10-10β,10-10β)	(-80.5575,10-10β,10-10β)
5	(0.001,10-10β,10-10β)	(-80.5564,10-10β,10-10β)	(-80.5544,10-10β,10-10β)
6	(0.01,10-10β,10-10β)	(-80.5548,10-10β,10-10β)	(-80.2948,10-10β,10-10β)

From the above tables, we see that the exterior penalty fuzzy valued function method converges at the 6^{th} iteration. Hence the optimal solution for the FNLPP (16) is $\tilde{y}_1 = (8.3333, 10 - 10\beta, 10 - 10\beta)$, $\tilde{y}_2 = (3.3333, 10 - 10\beta, 10 - 10\beta)$ with $\max \tilde{f}(\tilde{y}) = (80.2948, 10 - 10\beta, 10 - 10\beta)$.

That is the optimal solution of the fuzzy nonlinear programming problem (16) is $\tilde{y}_1 = (-1.6667 + 10\beta, 8.3333, 18.3333 - 10\beta)$, $\tilde{y}_2 = (-6.6667 + 10\beta, 3.3333, 13.3333 - 10\beta)$ with $\tilde{f}(\tilde{y}) = (70.2948 + 10\beta, 80.2948, 90.2948 - 10\beta)$.

7 Result and Discussion

Table (9) and figure (1) depicts the fuzzy optimal solution of the FNLPP (12) for different values of β.

Table 9: Optimal solution for different values of $\beta \in [0, 1]$

β	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{f}(\tilde{y})$
0	$(-0.57143, 1.42857, 4.42857)$	$(-1.14286, 0.85714, 3.85714)$	$(5.59156119, 7.5956119, 10.5915612)$
0.25	$(-0.07143, 1.42857, 3.67857)$	$(-0.64286, 0.85714, 3.10714)$	$(6.09156119, 7.5956119, 9.8415612)$
0.5	$(0.42857, 1.42857, 2.92857)$	$(-0.14286, 0.85714, 2.35714)$	$(6.59156119, 7.5956119, 9.0915612)$
0.75	$(0.92857, 1.42857, 2.17857)$	$(0.35714, 0.85714, 1.60714)$	$(7.09156119, 7.5956119, 8.3415612)$
1	$(1.42857, 1.42857, 1.42857)$ $= 1.42857$	$(0.85714, 0.85714, 0.85714)$ $= 0.85714$	$(7.59156119, 7.59156119, 7.59156119)$ =7.59156119

For the same fuzzy nonlinear programming problem (12), Akrami et al. (2016) obtained the

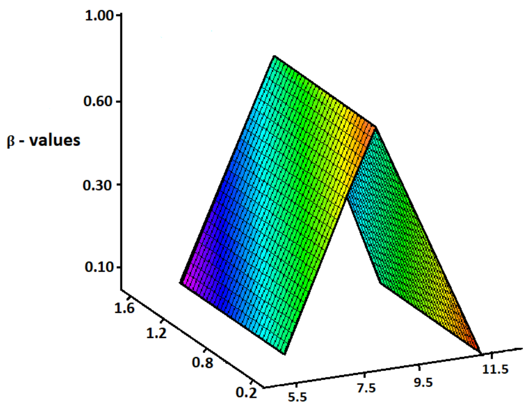

Figure 1: Optimal solution for different values of $\beta \in [0, 1]$

fuzzy optimal solution as $\tilde{y}_1 = (0, 1.429, 11)$, $\tilde{y}_2 = (0.6, 0.85714, 3)$ with $\tilde{f}(\tilde{y}) = (0.36, 7.597, 511)$. We see that the solution obtained by the proposed method is sharper than the solution obtained by Akrami et al. (2016).

Table (10) and figure (2) depicts the fuzzy optimal solution of the FNLPP (16) for different values of β.

Table 10: Optimal solution for different values of $\beta \in [0, 1]$

β	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{f}(\tilde{y})$
0	$(-1.6667, 8.3333, 18.3333)$	$(-6.6667, 3.3333, 13.3333)$	$(70.2948, 80.2948, 90.2948)$
0.25	$(0.8333, 8.3333, 15.8333)$	$(-4.1667, 3.3333, 10.8333)$	$(72.7948, 80.2948, 87.7948)$
0.5	$(3.3333, 8.3333, 13.3333)$	$(-1.6667, 3.3333, 8.3333)$	$(75.2948, 80.2948, 85.2948)$
0.75	$(5.8333, 8.3333, 10.3333)$	$(0.8333, 3.3333, 5.8333)$	$(77.7948, 80.2948, 82.7948)$
1	$(8.3333, 8.3333, 8.3333)$ $= 8.3333$	$(3.3333, 3.3333, 3.3333)$ $= 3.3333$	$(80.2948, 80.2948, 80.2948)$ =80.2948

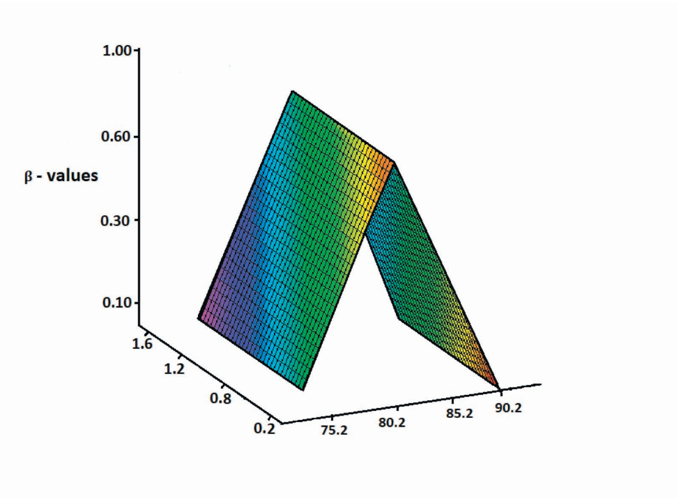

Figure 2: Optimal solution for different values of $\beta \in [0, 1]$

Himaja and Co. should plan for $\tilde{y}_1 = (-1.6667+10\beta, 8.3333, 18.3333-10\beta)$, $\tilde{y}_2 = (-6.6667+10\beta, 3.3333, 13.3333 - 10\beta)$ appearances of one minute advertisements in Raja Cable and Srija

Channel respectively to achieve its maximum total reach of $\tilde{f}(\tilde{y}) = (70.2948+10\beta, 80.2948, 90.2948-10\beta)$.

From the above tables (9), (10) and figures (1), (2), it can be seen that the proposed method provides flexibility to the decision maker to choose his /her desired solution by suitably selecting β.

8 Conclusion

In this paper, we introduced a new approach to solve fuzzy nonlinear programming problems using triangular fuzzy numbers. These numbers are represented with a location index, left fuzziness index, and right fuzziness index. We utilize the Exterior penalty fuzzy valued function method to obtain the fuzzy optimal solution without the need for conversion to crisp nonlinear programming. A numerical example shows that the proposed method produces a solution with less vagueness than existing methods. We also discussed an application of FNLPP in advertising sector. The decision maker can also choose a preferred solution by selecting a suitable value of $\beta \in [0, 1]$ depending on the situation and their own preferences.

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